The normalization of the scattering matrix

The main purpose of this section is to clarify the meaning of the scattering matrix elements tabulated in the database.

Let us start with considering the following simple scattering experiment [see Fig. 1]. A parallel beam of light is incident on a cloud (ensemble) of randomly oriented particles that scatter the incoming quasi-monochromatic light independently in all directions and without changing the wavelength. Particles and their mirror particles occur in equal numbers. Hence the cloud constitutes a macroscopically isotropic medium with mirror symmetry (see [1, section 2.7]). We assume the cloud to be optically thin so that multiple scattering can be neglected. The light scattered in a direction that makes an angle Θ (the scattering angle) with the direction of the incident beam is, in general, polarized. The properties of this scattered light, i.e. the flux and state of polarization, are measured by a detector at a distance, D, of the center of the cloud. The plane through the directions of incidence and scattering is called the scattering plane and we use this plane as a plane of reference for defining Stokes parameters as in Van de Hulst [2], Hovenier and Van der Mee [3] and Hovenier et al. [1]. The incident parallel beam of light is characterized by a flux vector $\pi\mathbf{\Phi}_0$. This is a column vector with four elements that are Stokes parameters in such a way that the first element is the net flux (irradiance) expressible in $\mathrm{Wm}^{-2}\mathrm{Hz}^{-1}$ or similar units. Similarly the flux vector of the light falling on the detector is the column vector $\pi\mathbf{\Phi}$, the first element of which is the net flux detected at a distance

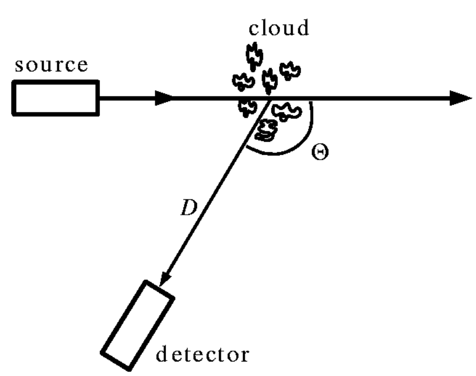

Fig. 1. A light source emits a parallel beam of quasi-monochromatic light. This light is scattered by a cloud of small randomly oriented particles in air or a liquid. A detector at a distance, D, of the cloud receives light scattered in a direction given by the scattering angle, Θ, and measures its net flux and state of polarization. The detector can be moved along a circle with radius D within a certain range of scattering angles.

D.

We can describe the simple experiment by means of the relation [2]

$$\mathbf{\Phi}(\lambda,\Theta) = \frac{\lambda^2}{4\pi^2 D^2}\mathbf{F}(\lambda,\Theta)\mathbf{\Phi}_0(\lambda), \tag{1}$$

where λ is the wavelength of the light and $\mathbf{F}(\lambda,\Theta)$ is a 4×4 scattering matrix of the cloud, which depends not only on λ and Θ but also on the number and properties of the scattering particles that contribute to the detected radiation. Another way of describing the simple experiment is by means of the fundamental formula [1, section 2.7]

$$\mathbf{\Phi}(\lambda,\Theta) = \frac{N^t\overline{C}_{sca}(\lambda)}{4\pi D^2}\mathbf{F}^{au}(\lambda,\Theta)\mathbf{\Phi}_0(\lambda). \tag{2}$$

Here N^t is the total number of scattering particles that contribute to the detected radiation, $\overline{C}_{sca}(\lambda)$ is their average scattering cross section per particle (average over all N^t particles and orientations) and $\mathbf{F}^{au}(\lambda,\Theta)$ is a scattering matrix that is normalized so that the average over all directions of the 1-1 element equals unity. The superscript au is used to indicate this normalization. The 16 elements of $\mathbf{F}(\lambda,\Theta)$ and $\mathbf{F}^{au}(\lambda,\Theta)$ are indicated as $F_{i,j}(\lambda,\Theta)$ and $F^{au}_{i,j}(\lambda,\Theta)$, respectively,where $i,j = 1,2,3,4$. Hence we have

$$\frac{1}{4\pi}\int\limits_{(4\pi)} d\Omega\, F^{au}_{1,1}(\lambda,\Theta) = 1, \tag{3}$$

where $d\Omega$ is an element of solid angle. Eq. (3) can also be written as

$$\frac{1}{2}\int\limits_0^\pi d\Theta \sin\Theta\; F^{au}_{1,1}(\lambda,\Theta) = 1. \tag{4}$$

It should be noted that $\mathbf{F}(\lambda,\Theta)$ and $\mathbf{F}^{au}(\lambda,\Theta)$ are both called scattering matrices although their normalizations are quite different. Comparing Eqs. (1) and (2) yields the following relationship between these two scattering matrices

$$\mathbf{F}(\lambda,\Theta) = \frac{\pi N^t}{\lambda^2}\overline{C}_{sca}(\lambda)\mathbf{F}^{au}(\lambda,\Theta). \tag{5}$$

The normalization of $\mathbf{F}(\lambda,\Theta)$ is now readily obtained from Eqs. (4)-(5). The result is

$$\frac{1}{2}\int_0^{\pi} d\Theta \sin\Theta \; F_{1,1}(\lambda,\Theta) = \frac{\pi N^t}{\lambda^2}\overline{C}_{sca}(\lambda). \tag{6}$$

It follows from Eqs. (1)-(2) that all elements of $\mathbf{F}(\lambda,\Theta)$ and $\mathbf{F}^{au}(\lambda,\Theta)$ are dimensionless. Several other scattering matrices occur in the literature. Differences in normalization are quite common and require special attention. The same is true for scattering (phase) functions which are the 1-1 elements of scattering matrices.

An advantage of $\mathbf{F}(\lambda,\Theta)$ is its simple additivity property. This means that if a cloud consists of a mixture of G different subclouds of particles we have for the cloud

$$\mathbf{F}(\lambda,\Theta) = \sum_{g=1}^{G} \mathbf{F}_g(\lambda,\Theta), \tag{7}$$

where the index g is used to indicate the subclouds. Here the summation can be replaced by integration over a continuous variable characterizing e. g. the size or shape of the particles. Combining Eqs. (5) and (7) we see that the matrices $\mathbf{F}_g^{au}(\lambda,\Theta)$ of subclouds cannot simply be added to get the corresponding matrix of a cloud. Instead we have

$$\mathbf{F}^{au}(\lambda,\Theta) = \frac{\sum_{g=1}^{G} N_g^t[\overline{C}_{sca}(\lambda)]_g[\mathbf{F}^{au}(\lambda,\Theta)]_g}{\sum_{g=1}^{G} N_g^t[\overline{C}_{sca}(\lambda)]_g}, \tag{8}$$

where N_g^t is the total number of particles of subcloud g that contribute to the detected radiation and $[\overline{C}_{sca}(\lambda)]_g$ is their average scattering cross section per particle. On the other hand $\mathbf{F}^{au}(\lambda,\Theta)$ has the advantage that Eq. (2) shows explicitly how the flux vector of the scattered light at a distance D depends on the number of participating particles and on their average scattering cross section. $\mathbf{F}(\lambda,\Theta)$ and similar scattering matrices are often used in studies of single light scattering [2; 4], whereas scattering matrices like $\mathbf{F}^{au}(\lambda,\Theta)$, applied to an ensemble of particles in a unit volume, are usually preferred in multiple scattering studies [1; 5; 6; 7].

If the incident light is unpolarized only the first element of $\pi\mathbf{\Phi}_0(\lambda)$ is nonzero. By measuring the net flux of the scattered light for one and the same cloud at various values of the scattering angle, but the same values of λ and D and the same net flux of incident unpolarized light, we can readily determine from Eqs. (1)-(2) the ratio

$$\frac{F_{1,1}(\lambda,\Theta)}{F_{1,1}(\lambda,\Theta_0)} = \frac{F_{1,1}^{au}(\lambda,\Theta)}{F_{1,1}^{au}(\lambda,\Theta_0)}, \tag{9}$$

where Θ_0 is a fixed scattering angle. Hence, for a fixed value of λ both scattering functions can readily be measured in a relative sense. This means, loosely speaking, that we can measure scattering functions that are normalized to unity at a scattering angle of, for instance, 30 degrees. Note that no knowledge of N^t and $\overline{C}_{sca}(\lambda)$ is needed for this purpose, but care should be taken to keep their product constant while varying the scattering angle or to perform corrections for time variations.

The simple experiment can be continued by changing the state of polarization of the incident beam and analyzing the state of polarization of the scattered radiation. In this way we can obtain the 15 ratios

$$\frac{F_{i,j}(\lambda,\Theta)}{F_{1,1}(\lambda,\Theta)}=\frac{F^{au}_{i,j}(\lambda,\Theta)}{F^{au}_{1,1}(\lambda,\Theta)} \tag{10}$$

for a fixed value of λ and for all values of $i,j=1,2,3,4$, except $i=j=1$. Hence the matrix elements with these values of i and j are not determined themselves, but only their values relative to the scattering function at the same scattering angle and wavelength. Knowledge of N^t and $\overline{C}_{sca}(\lambda)$ is not necessary for this.

The simple experiment described above contains the basic features of the more sophisticated experiments that yielded the numbers of the database for a variety of samples. For a more detailed description of these experiments see [8; 9; 10]. Experimental checks on neglecting multiple scattering and the random orientation of the particles have been described in [9]. Thus we obtained the ratios given by Eqs. (9) and (10), where Θ_0 is 30 degrees for all measurements of aerosol particles. For greater clarity the headings of the columns containing numbers for $F_{1,1}(\lambda,\Theta)/F_{1,1}(\lambda,30°)$ in the database are now `F11/F11(30deg)` for all aerosol particles and no longer `F11` as they were in the version of the database discussed by Volten et al [11]. The headings of the columns containing numbers for the ratios of different elements of the scattering matrix, as shown in Eq. (10), have not been changed. The hydrosol $F_{11}(\Theta)$ functions have been normalized slightly differently from the aerosol $F_{11}(\Theta)$ functions, i.e. they have been scaled at a scattering angle of 90 degrees to a standard function measured for San Diego Harbor water [15], which is indicated as `F11/SDH90` in the database. To obtain the same normalization as for the aerosol particles, it suffices to divide all values of F_{11}/SDH90 of a hydrosol by its value at 30 degrees. The values of $-F_{12}(\Theta)/F_{11}(\Theta)$ remain the same.

The range of scattering angles in the database is limited to at most 3-174 degrees. Several methods have been developed to extrapolate the data to the full range, in particular for the relative scattering function as given by Eq. (9) (see e.g. [12; 13; 14]). After such an extrapolation has been performed we can first obtain $F^{au}_{1,1}(\lambda,\Theta_0)$ by integration since

$$\frac{1}{2}\int_0^\pi d\Theta\sin\Theta\,\frac{F_{1,1}(\lambda,\Theta)}{F_{1,1}(\lambda,\Theta_0)}=\frac{1}{F^{au}_{1,1}(\lambda,\Theta_0)}, \tag{11}$$

where Eqs. (4) and (9) have been used. Eq. (9) then provides $F^{au}_{1,1}(\lambda,\Theta)$, i.e. the scattering function with the normalization given by Eq. (4), instead of only its ratio to the value at 30 degrees. Finally, the extrapolated values of the ratios given by Eq. (10) can be employed to obtain $F^{au}_{i,j}(\lambda,\Theta)$ for all values of i, j and Θ. The extrapolations to the full range of scattering angles are especially important for multiple scattering computations with or without taking polarization into account. Values of the scattering function of the Synthetic Average Volcanic Scattering Matrix refer to $F^{au}_{1,1}(\lambda,\Theta)$ and are indicated in the database as `F11_au`.

So far we have considered light scattering by one and the same cloud of particles at various scattering angles but at the same wavelength. Let us now look at data in the database for the same substance at different wavelengths. If we introduce

$$f(\lambda,\Theta) = \frac{F_{1,1}(\lambda,\Theta)}{F_{1,1}(\lambda,\Theta_0)} \tag{12}$$

it is clear that this function equals unity at Θ_0 for all wavelengths, but generally changes with wavelength for other scattering angles. If for some range of scattering angles

$$f(\lambda_1,\Theta) > f(\lambda_2,\Theta) \tag{13}$$

we can rewrite this as (cf. Eq. (9))

$$F^{au}_{1,1}(\lambda_1,\Theta) > \left[\frac{F^{au}_{1,1}(\lambda_1,\Theta_0)}{F^{au}_{1,1}(\lambda_2,\Theta_0)}\right] F^{au}_{1,1}(\lambda_2,\Theta). \tag{14}$$

Since the expression between square brackets in this inequality was not measured, it is not tabulated in the database and we cannot conclude from Eq. (14) that $F^{au}_{1,1}(\lambda,\Theta)$ is larger or smaller at λ_1 than at λ_2 in the range of scattering angles considered. However, by extrapolation we can find $F^{au}_{1,1}(\lambda_1,\Theta)$ as well as $F^{au}_{1,1}(\lambda_2,\Theta)$. In this way we can compare the values at two wavelengths for the same scattering angles. The situation is different for the other scattering matrix elements. Indeed if $F_{i,j}(\lambda,\Theta)/F_{1,1}(\lambda,\Theta)$ is larger at one wavelength than at another the same is true for $F^{au}_{i,j}(\lambda,\Theta)/F^{au}_{1,1}(\lambda,\Theta)$, as follows from Eq. (10).

By way of example we consider scattering of (unpolarized) sunlight by particles of a comet. We neglect multiple scattering and assume the particles to be in random orientation with equal amounts of particles and their mirror particles. Even if we were to know $F^{au}_{1,1}(\lambda_1,\Theta)$ to be larger than $F^{au}_{1,1}(\lambda_2,\Theta)$ we would not know if the observed flux $\pi\Phi_1(\lambda_1,\Theta)$ is larger than $\pi\Phi_1(\lambda_2,\Theta)$, since this also depends on the wavelength dependence of the average scattering cross sections, the number of participating particles at the time of the observations and the incident flux. But it follows from Eqs (10) and (2) that, if e.g. $F_{2,1}(\lambda,\Theta)/F_{1,1}(\lambda,\Theta)$ is larger at one wavelength than at another, this also holds for $\Phi_2(\lambda, \Theta)/\Phi_1(\lambda, \Theta)$, where the subscripts 1 and 2 refer to the first and second element, respectively, of the observed column vector $\mathbf{\Phi}$. This means that the numbers of the database can be used directly for interpreting observations of the wavelength dependence ("color") of the polarization, but not for comparison with the wavelength dependence (color) of the observed brightness.

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